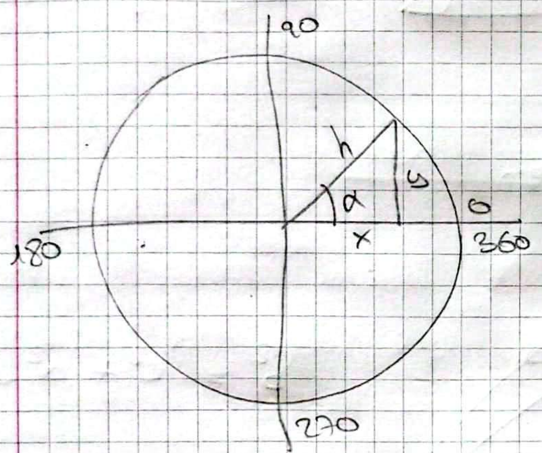
Calculus

Trigonometry

- Trigonometric definitions

in a circle of radious h



$$\boxed{\sin\alpha = \frac{y}{h} \qquad \cos\alpha = \frac{x}{h} \qquad tg\,\alpha = \frac{y}{x}}$$

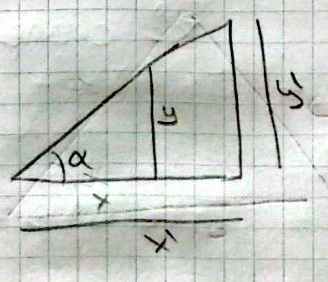
$\sin^2\alpha + \cos^2\alpha = 1$ ← trigonometric identity

- radians to angles

π rad $= 180°$ $\qquad$ $\pi/2$ rad $= 90°$

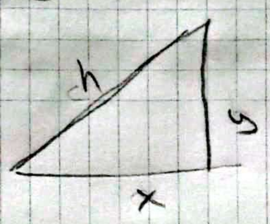
2π rad $= 360°$ $\qquad$ 0 rad $= 0°$

- talen theorem



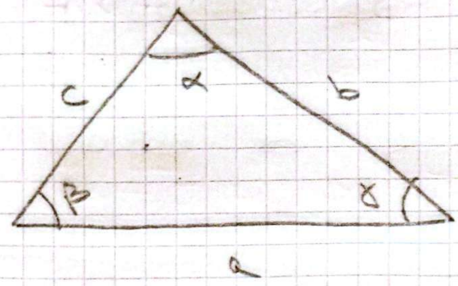
$tg\,\alpha = \frac{y}{x} = \frac{y'}{x'}$

- Pitagoran theorem



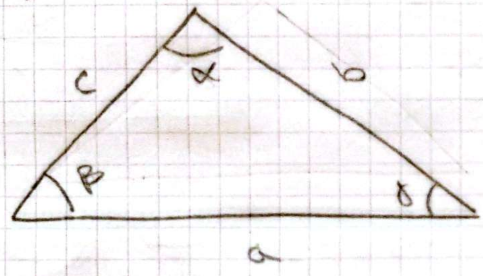
$h^2 = x^2 + y^2$

- sinus theorem



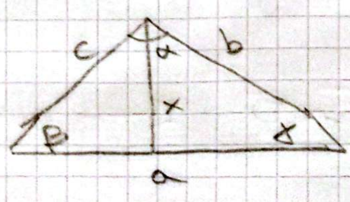
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

- cosinus theorem



$$b^2 = c^2 + a^2 - 2\,c\cdot a\cos\beta$$

problem: prove the sin theorem using the trigonometrie identities



$\sin\beta = \frac{x}{c}$ $\quad$ $\sin\gamma = \frac{x}{b}$

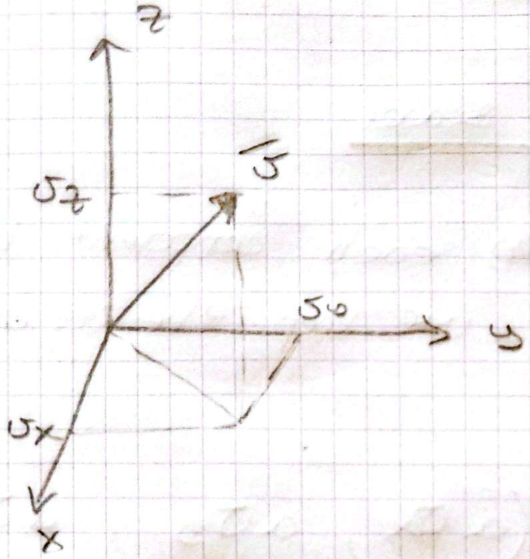
$x = c\cdot\sin\beta$ $\quad$ $x = b\cdot\sin\gamma$

$$c\sin\beta = b\sin\gamma$$

$$\frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Vectors:

a vector $\bar{v}$ is a mathematical object caracterized by a direction $\hat{v}$ and a modulus



$\bar{v} = (v_x, v_y, v_z)$

it can be represent as a list of components each component a dimension

modul: $|\bar{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Direction: $\hat{v} = \dfrac{\bar{v}}{|\bar{v}|} = \dfrac{(v_x, v_y, v_z)}{\sqrt{v_x^2 + v_y^2 + v_z^2}}$

a unitary vector is the one that has modulus 1

problem: find the modulus and direction of the vector $\bar{v} = (3, 2, 8)$

- **sum of vectors**

we sum each component seperated and we keep the vectorial structure

$$\bar{u} + \bar{v} = (u_x + v_x , u_y + v_y , u_z + v_z)$$

- **product by scalar**

we multiply each component by the scalar and we keep the vectorial structure

$$a \cdot \bar{v} = (a v_x , a v_y , a v_z)$$

- **scalar product of two vectors**

it is a matricial operation and results in a scalar value

$$\rightarrow \bar{v} \cdot \bar{u} = (v_x , v_y , v_z) \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} =$$

$$= v_x \cdot u_x + v_y \cdot u_y + v_z \cdot u_z$$

$$\rightarrow \bar{v} \cdot \bar{u} = |\bar{v}| \cdot |\bar{u}| \cdot \cos \alpha$$

where α is the angle between the two vectors

• <u>properties of scalar product</u>

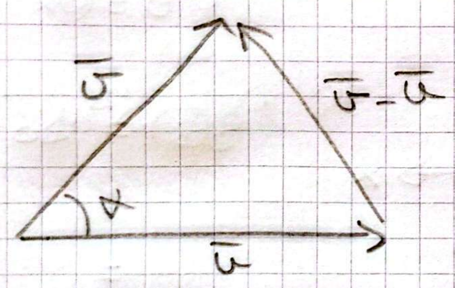
• if two vectors are parallel so

$u = a\bar{v}$

• if two vectors are orthogonal

$\bar{u} \cdot \bar{v} = 0$

→ problem : show the cos theorem using the scalar product relations



?

$|\bar{v} - \bar{u}| = |\bar{u}|^2 + |\bar{v}|^2 - 2|\bar{u}||\bar{v}|\cos$

$$2 \cdot |\bar{v}| \cdot |\bar{u}| \cos\alpha = |\bar{u}|^2 + |\bar{v}|^2 - |\bar{v} - \bar{u}|$$

$$= \sqrt{u_x^2 + u_y^2}^{\,2} + \sqrt{v_x^2 + v_y^2}^{\,2} - \sqrt{(v_x - u_x)^2 + (v_y - u_y)^2}^{\,2}$$

$$= \cancel{u_x^2} + \cancel{u_y^2} + \cancel{v_x^2} + \cancel{v_y^2} - (\cancel{v_x^2} + \cancel{u_x^2} - 2v_x u_x) - (\cancel{v_y^2} + \cancel{u_y^2} - 2v_y u_y)$$

$$= +2\,v_x u_x + 2\,v_y u_y$$

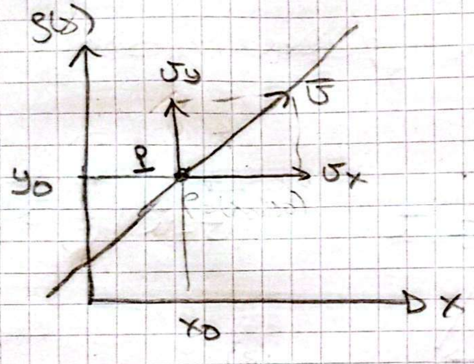
$$\underbrace{|\bar{v}| \cdot |\bar{u}| \cdot \cos\alpha}_{\bar{v} \cdot \bar{u}} = \underbrace{v_x u_x + v_y u_y}_{\bar{v} \cdot \bar{u}}$$

Analisis of functions

A function is a mathematical object that produce a unic output $f(x)$ given an input.

In 2 dimensions functions are represented an curbe

- Liner: $f(x) = a + bx$



given a point and a vector in the space

$\bar{v} = (v_x, v_y)$ $\quad P = (x_0, y_0)$

$$f(x) = y_0 + \frac{v_y}{v_x}(x - x_0)$$

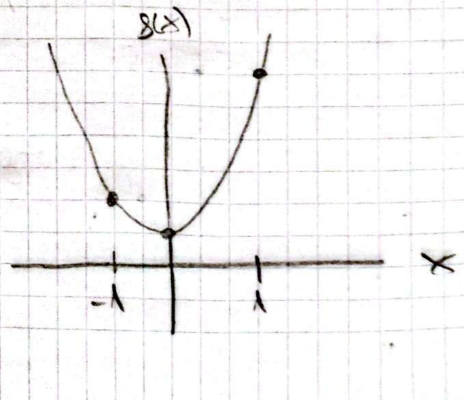
↑ slope of the line

- Parabolan: $f(x) = a + bx + cx^2$

a parabola is a polinomium of second order

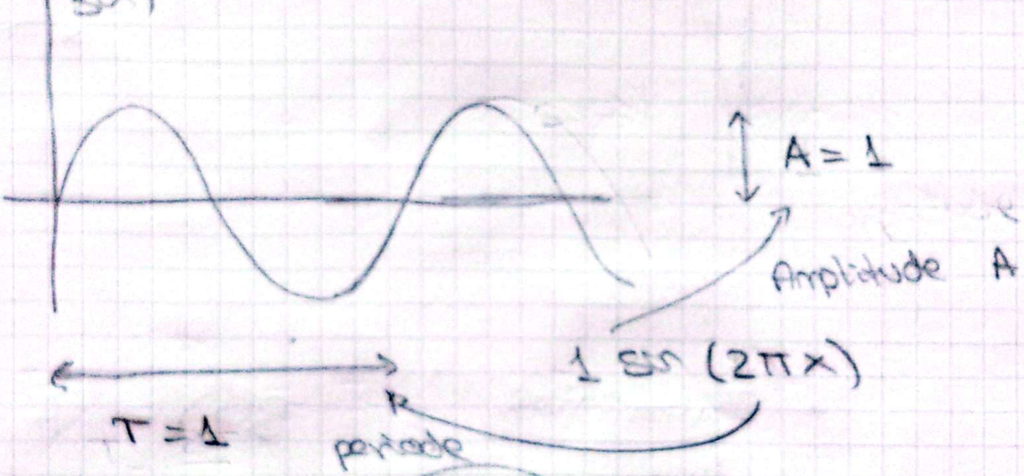
exemple: $f(x) = 1 + 2x + 3x^2$

$f(x)$	x
1	0
6	1
2	-1



• trigonometrical
$$\begin{cases} f(x) = A\sin(Bx) \\ f(x) = A\cos(Bx) \\ f(x) = A\tan(Bx) \end{cases}$$

trigonometrical functions are periodic (they repeat themselves in order)

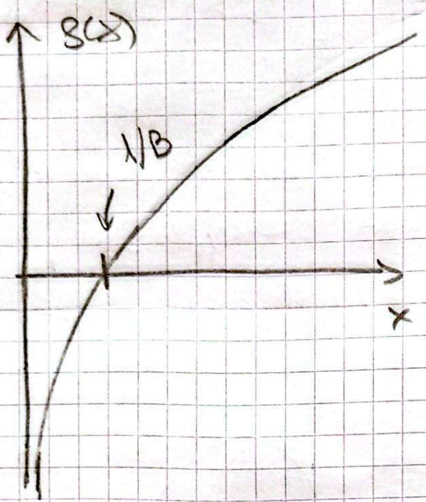


$$T = \frac{2\pi}{|B|}$$

problem: plot the function $f(x) = 3\cos(2x)$
determine the amplitude A and periode T

problem: use geogebra to plot the function
$f(x) = tg(2\pi x)$ find the discontinuities, are them recursives?

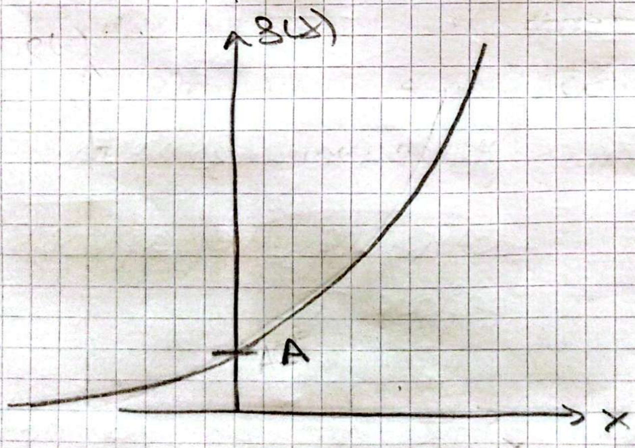
- Logaritmical $f(x) = A \log(Bx)$



A log is just defined in positive x range.

- log never reach 0
- log tend to ∞ for $x \to \infty$

- Exponential : $f(x) = A \exp(Bx)$



an exponential is never negative

- exp tends to 0 for $x \to -\infty$
- exp tends to ∞ for $x \to \infty$

problem : use geogebra for representing the functions explained and play with the parameters

Limits

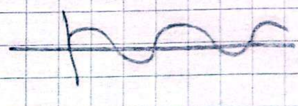
the limit of a function is the value that the function takes when we approach a given point

$$\lim_{x \to x_0} f(x) = \begin{cases} \nexists \\ \text{constant or } 0 \to \text{horizontal assimptot (the function converge)} \\ \pm\infty \begin{cases} \pm\infty \\ \text{vertical assimptota} \\ \text{obliguous assiptota} \end{cases} \end{cases}$$

- The limit does not exist when the function does not tend to a determined value

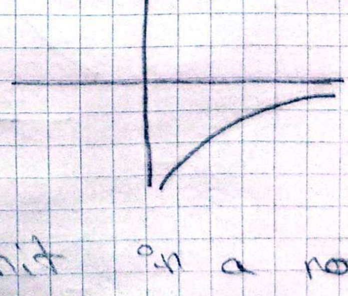
example:

$$\lim_{x \to \infty} \sin(x) = \nexists$$



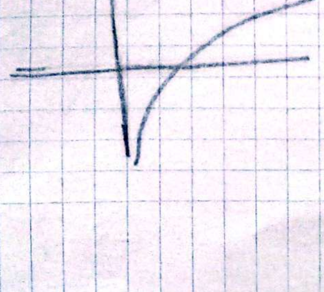
- the limit tends to a constant or 0 in this case we say the function converge and the aproximation is horizontaly assiptotic

$$\lim_{x \to \infty} 1/x = 0$$



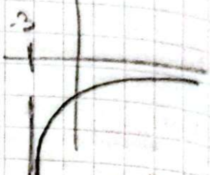
- the limit tends to infinit in a not linear way and is neither a vertical assiptot

$$\lim_{x \to \infty} \ln(x) = \infty$$



• the limit tends to ingenit in a vertical assimptot : in this case we can identify because the function generates a discontinuity of the kind $\frac{1}{0}$

$$\lim_{x \to -3} \left(\frac{1}{x+3} \right) = \frac{1}{-3+3} = \frac{1}{0} \; ?$$

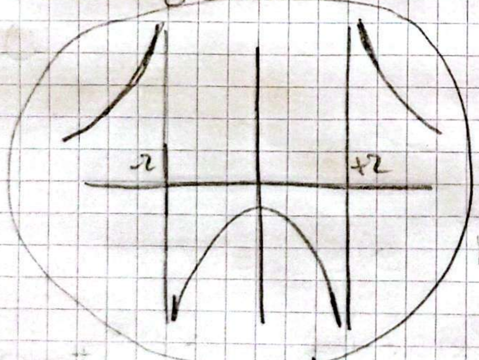


vertical assimptot

in this case we now wander wether the assimptot es positive or negative we can know by checking the right and lest limit

exempe $g(x) = \frac{1}{x^2-4}$

discontinuities in $x^2-4=0 \Rightarrow x = \pm 2$



$$\lim_{x \to \pm 2} \frac{1}{x^2-4} = \frac{1}{0}$$

+2

$\oplus$ $\lim_{x \to 2^+} \frac{1}{x^2-4} = \frac{1}{2{,}001^2-4} = +\infty$

$\ominus$ $\lim_{x \to 2^-} \frac{1}{x^2-4} = \frac{1}{1{,}999^2-4} = -\infty$

-2

$\oplus$ $\lim_{x \to -2^+} \frac{1}{x^2-4} = \frac{1}{(-1{,}999)^2-4} = -\infty$

$\ominus$ $\lim_{x \to -2^-} \frac{1}{x^2-4} = \frac{1}{(-2{,}001)^2-4} = +\infty$

- the limit tends to ingini in a oblique assimptot

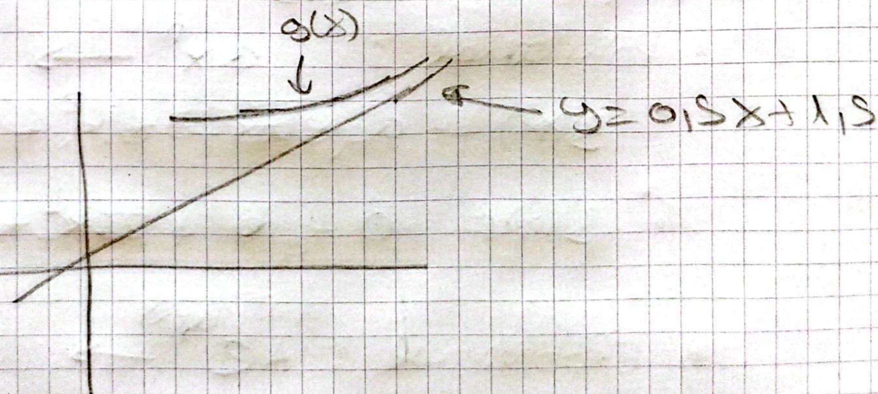
↓

in this case the limit tends to ∞ but in a linear way. It happens when the numerator is one order bigger than the denominator

$$g(x) = \frac{x^2}{2x-6}$$

$$\lim_{x \to \infty} \frac{g(x)}{x} = 0{,}5$$ ← oblique assiptot slope

$$\lim_{x \to \infty} \left[g(x) - 0{,}5x \right] = 1{,}5$$ ← oblique assiptot constant

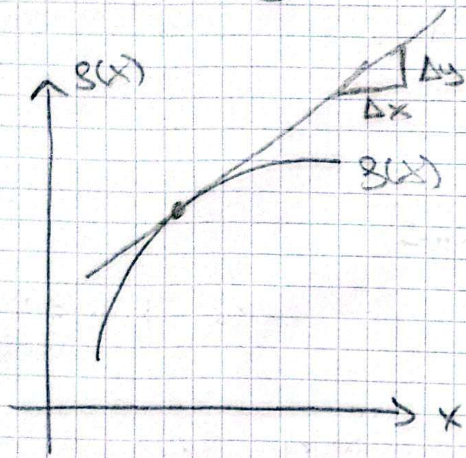


$y(x=100) = 51{,}50$

$g(x=100) = 51{,}54$ ← bigger

Derivatives

it is the slope of a function. the derivative defines how a function changes in a given dimension.



$$f'(x) = \frac{d f(x)}{dx} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

instead of computing the limit every time we can use derivative tables

→ simple functions

- Polinomial : $a x^n \longrightarrow a n \cdot x^{n-1}$

$$f(x) = a + bx + cx^2 + dx^3$$

$$f'(x) = 0 + b + 2cx^1 + 3dx^2$$

- Exponential : $Ae^{u(x)} \longrightarrow u'(x) \cdot Ae^{u(x)}$

$$f(x) = 3e^{2+4x}$$

$$f'(x) = 3(2+4x)' \cdot e^{2+4x} = 3 \cdot 4 e^{2+4x}$$

- logarithmic $\log_a(u) = u' \cdot \frac{1}{u \ln(a)}$

$$f(x) = \ln(3x)$$

$$f'(x) = 3 \cdot \frac{1}{3x \ln(e)}$$

- Power $a^u \to a^u \ln(a) u'$

$$f(x) = a^{3x+2}$$

$$f'(x) = a^{3x+2} \cdot \ln(a) \cdot 3$$

- sin and cos

$$\sin(u) \longrightarrow \cos(u)\, u'$$
$$\cos(u) \longrightarrow -\sin(u)\, u'$$

$$f(x) = 3\sin(2x) + 4\cos(5x^2+3)$$

$$f'(x) = 3\cos(2x)\cdot 2 + 4(-)\sin(5x^2+3)\cdot 10x$$

→ composed functions

- multiplication of two functions $\quad u\cdot v \longrightarrow u'v + u\cdot v'$

$$f(x) = (3x+5)\sin(2x)$$

$$f'(x) = (3x+5)'\cdot\sin(2x) + (3x+5)(\sin(2x))'$$
$$= 3\cdot\sin(2x) + (3x+5)\cdot\cos(2x)\cdot 2$$

- division of two functions $\quad \frac{u}{v} \longrightarrow \frac{u'\cdot v - u\cdot v'}{v^2}$

$$f(x) = \frac{\cos(5x)}{3e^{2x}}$$

$$f'(x) = \frac{(\cos(5x))'\cdot 3e^{2x} - \cos(5x)\cdot(3e^{2x})'}{[3e^{2x}]^2}$$

$$= \frac{-\sin(5x)\cdot 5\cdot 3e^{2x} - \cos(5x)\, 3\cdot 2e^{2x}}{9e^{4x}}$$

Problem: compute the following derivative

$$f(x) = \sqrt{3x} \cdot \log_{10}(4x^2)$$

$$f(x) = a^{(4x+2)} \cdot tg(3x)$$

$$f(x) = 3[2x + 8\sin(3x)] \cdot \sqrt[3]{2x+1}$$

Problem: Knowing that the velocity $v(t)$ the first time derivative of the position $x(t)$. And the acceleration is the second time derivative of the position. Find the expression of the velocity and acceleration for an object that follow the movement

$x(t)$

$$v(t) = \frac{d\,x(t)}{dt}$$

$$a(t) = \frac{d^2\,x(t)}{dt^2}$$

$$x(t) = (3t^2 + 2t)(5t + 1)$$

- where we find the maximum velocity? and the minimum?
- Does it have any maximum acceleration?
- Represent $x(t)$ $v(t)$ and $a(t)$ you can use geogebra

Aplication of derivatives

→ l'Hôpital rule : we can compute limits of composed functions using derivatives if the limit is 0/0 kind

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

example

- $\lim_{x \to 1} \frac{1 - x^2}{\sin(\pi x)} = \frac{0}{0}$

$= (\text{l'hopital}) = \lim_{x \to 1} \frac{2x}{\cos(\pi x)} = \frac{2}{1} = 2$

- $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{0}{0}$

$(\text{l'hopital}) = \lim_{x \to 0} \frac{\sin(x)}{2x} = \frac{0}{0}$

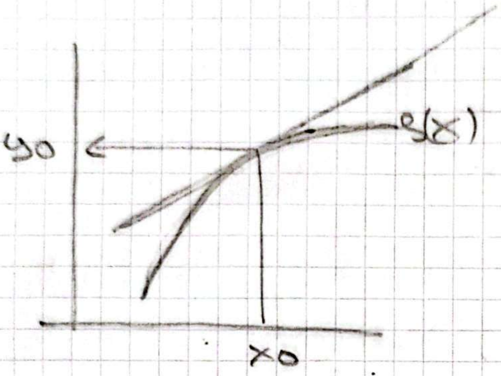
$(\text{l'hopital}) = \lim_{x \to 0} \frac{\cos(x)}{2} = 1/2$

→ tangent line to a curve in a point

↓

since the derivative is the slope of the function

$$\boxed{(y-y_0) = m(x-x_0)}$$



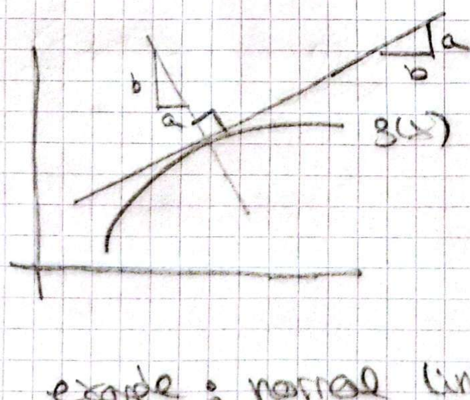
example: tangent line of $f(x) = 3x+2x^2$ at $x_0 = 1$

$x_0 = 2$ $\quad y_0 = f(x_0) = 3\cdot1+2\cdot1^2 = 5$

$m = f'(x_0) = 3+2\cdot2\cdot x_0 = 3+4\cdot1 = 7$

$(y-5) = 7(x-2)$ ← tangent line

→ normal line to a curve in a point



$$(y-y_0) = -\frac{1}{m}(x-x_0)$$

example: normal line of $f(x) = 3x+2x^2$ at x_0

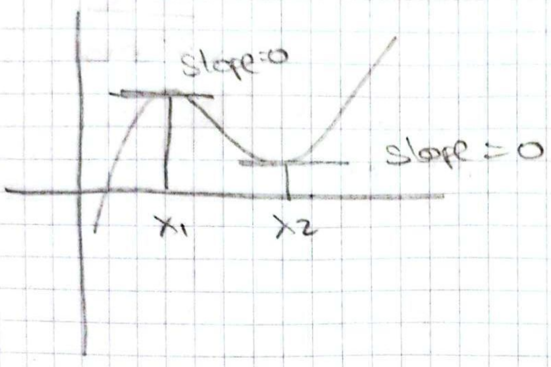
$x_0 = 1$ $\quad y_0 = 5$ $\quad -1/m = -1/7$

$(y-5) = -\frac{1}{7}(x-2)$ ← normal line

→ relative extrems of a function

this is the maxim and minim points

they are thiscores that satisfy slope = 0



It means the derivative at x_1 and x_2 must be zero

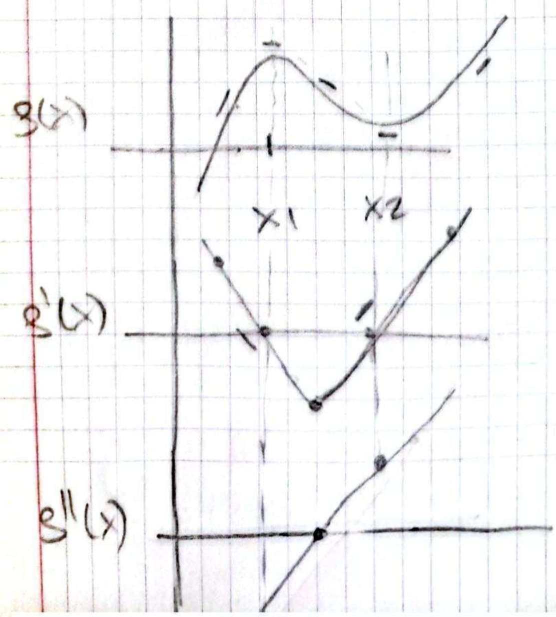
maximum: also satisfy that second derivative is negative

$$f'(x_1) = 0 \quad \& \quad f''(x_1) < 0$$

minimum: also satisfy that second derivative is positive

$$f'(x_2) = 0 \quad \& \quad f''(x_2) > 0$$

why ?



in max the slope is 0

the slope of the slope is negative

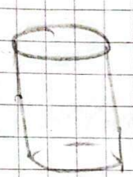
inflexion points has second derivative zero

$$f''(x_{inf}) = 0$$

→ optimitzation

optimization problems are thisones that can be solved by ginding the minimum or maximum of a given gunction

Exemple 1: minimum surgace required to envelop a cylinder of gixed volum



$$V = \pi r^2 \cdot h$$

$$A = 2\pi r \cdot h + 2\pi r^2$$

girst we write the area with one single variable

$$A = 2\pi r \cdot \left(\frac{V}{\pi r^2}\right) + 2\pi r^2 = \frac{2V}{r} + 2\pi r^2$$

we digerenciate the area

$$A' = 2V\left(\frac{-1}{r^2}\right) + 2\pi\, 2r$$

ginally we equal to zero to gind the minimum

$$A' = 0 = -\frac{2V}{r^2} + 4\pi r \Rightarrow 4\pi r = \frac{2V}{r^2}$$

$$r = \sqrt[3]{\frac{V}{2\pi}}$$

ginally we get the h

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi \sqrt[3]{\frac{V}{2\pi}}^{\,2}}$$

Maximazing profit in a manufacturing

exope 2: a company produce and sells products

the revenue R(x) and the cost C(x) asociated with producing x units of product are

$$R(x) = 100x - 0{,}5x^2$$

$$C(x) = 20x + 1000$$

the profit is the difference betwen revenue and cost

$$P(x) = R(x) - C(x)$$

find the number of units that maximize the profit

$\boxed{\text{solution: } 80 \text{ units}}$

Partial derivatives

partial derivatives arise in functions with more than one variable

$$f(x, y, z, ,...)$$

we perform a partial derivative by differenciating on one of the variables keeping the other constant

$\frac{\partial}{\partial x} f(x,y) \rightarrow$ partial derivative on x where y is treated as a constant

$\frac{\partial}{\partial y} f(x,y) \rightarrow$ partial derivative on y (x constant)

example : $f(x,y) = 3xy + 2x^2 \sin(y)$

$$\partial_x f = \frac{\partial f(x,y)}{\partial x} = 3y + 2\sin(y) \cdot 2x$$

$$\partial_y f = \frac{\partial f(x,y)}{\partial y} = 3x + 2x^2 \cos(y)$$

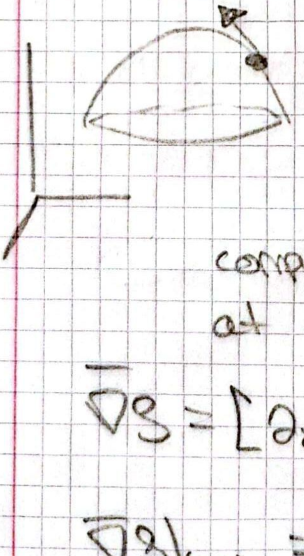
gradient

the gradient ∇ of a function f is a vecter with the partial derivatives en his components

$$\vec{\nabla} f = [\partial_x f, \partial_y f, \partial_z f]$$

the gradient vector evaluated in a coordenate (x_0, y_0, z_0) points to the direction of maximum increase of the function

exemple: leth imagin a function that desfines a given surface



$$f(x,y) = 2x^2y^2$$

point: $(x_0, y_0) = (3,2)$

compute the gradient and evaluate at the point

$$\vec{\nabla} f = [\partial_x f, \partial_y f] = [4xy^2, 4x^2y]$$

$$\vec{\nabla} f\big|_{(3,2)} = [4\cdot 3\cdot 2^2, 4\cdot 3^2\cdot 2] = [48, 72]$$

- direction of maximum decrease: can be extracted by inverting the sign of the gradien

max decrease $= -\vec{\nabla} f$

• Total slope of a gradient can be computed by extracting the lenght of the gradient vector

$$\overline{\nabla f} = (\partial_x f, \partial_y f, \partial_z f)$$

$$\text{slope} = |\overline{\nabla f}| = \sqrt{[\partial_x f]^2 + [\partial_y f]^2 + [\partial_z f]^2}$$

example : $\sqrt{48^2 + 72^2} = 86,5$

• the slope of the function in a vector direction can be computed projecting (scalar product) the gradient vector in the unitary direction of the vector

$$\text{slope in } \overline{v} \text{ direction} = \overline{\nabla f} \cdot \hat{v}$$

exemple : $f(x,y) = 2x^2 y^2$ in point $(3,2)$

$$\overline{v} = (3,2)$$

$$\overline{\nabla f} = [4xy^2, 4x^2y] = (48,72)$$

$$\hat{v} = \frac{(3,2)}{\sqrt{3^2+2^2}}$$

$$\text{slope in } v \text{ direction} = (48,72)\begin{pmatrix}3\\2\end{pmatrix}\frac{1}{\sqrt{13}} =$$

$$= \frac{1}{\sqrt{13}}(48 \cdot 3 + 72 \cdot 2) = 79,8$$

- maximum : is this point that satisfy the gradient given the vector $\overline{0}$ and the second derivatives are all negatives

$$\overline{\nabla f} = \overline{0} \quad \& \quad \overline{\nabla^2 f}\Big|_{max} < 0$$

- minimum : is this point that satisfy the gradient given the vector $\overline{0}$ and the second derivative are all positives

$$\overline{\nabla f} = \overline{0} \quad \& \quad \overline{\nabla^2 f}\Big|_{min} < 0$$

exemple $f(x,y) = 2x^2 + 3y^2$

$$\overline{\nabla f} = [\,4x\,,\,6y\,] = [0,0] \begin{cases} x = 0 \\ y = 0 \end{cases}$$

$$\overline{\nabla^2 f} = [\,4\,,\,6\,] > [0,0]$$

$\Downarrow$

it is a minimum in $(x,y) = (0,0)$

- saddle point : is this point that satisfy the second derivatives are all zero

$$\overline{\nabla^2 f} = \overline{0}$$

Hessian and saddle point

the hessian matrix is thisone that contain all the second partials derivatives of his components

$$H = \begin{pmatrix} \partial_{xx} f & \partial_{xy} f & \partial_{xz} f \\ \partial_{yx} f & \partial_{yy} f & \partial_{yz} f \\ \partial_{zx} f & \partial_{zy} f & \partial_{zz} f \end{pmatrix}$$

- is the determinant of $|H|$ evaluated at a point is biger then 0, we have a maxim or min

$|H| > 0 \Rightarrow$ maxim or a min

↳ max has also the diagonal terms < 0

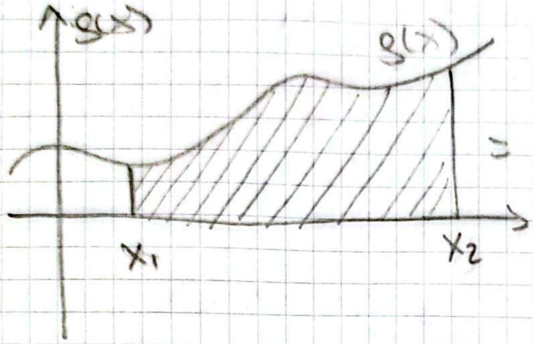
↳ min has also the diag terms > 0

- is the determinant of $|H|$ evaluated at a point is smaler then 0, we have a saddle point

$|H| < 0 \Rightarrow$ saddle point

Integration

A defined integral is this operation that compute the area under a curve delimitated by extrems x_1 and x_2 of integration



$$= \text{Area} = \int_{x_1}^{x_2} f(x)\, dx$$

Integration is the invers procedure to derivation. we can use the table of derivatives in inverse way

- Polinomial $f(x) = a + bx$

$$\int_{x_1}^{x_2} [a+bx]\, dx = \left[ax + \frac{bx^2}{2}\right]_{x_1}^{x_2} =$$

$$= \left[ax_2 + \frac{bx_2^2}{2}\right] - \left[ax_1 + \frac{bx_1^2}{2}\right]$$

problem : compute defined integral of

$f(x) = x^3 + 5x^4$ between $x_1 = 1$

and $x_2 = 3$

• Integral by parts : $\int u\,dv = uv - \int v\,du$

ex1: $\int x e^{-x} dx =$

$$\begin{bmatrix} u = x & \longrightarrow & du = dx \\ v = -e^{-x} & \longrightarrow & dv = +e^{-x} dx \end{bmatrix}$$

$$= -xe^{-x} - \int -e^{-x} dx =$$

$$= -xe^{-x} + \underbrace{\int e^{-x} dx}_{-e^{-x}} = -e^{-x}(x+1)$$

ex2: $\int x^2 e^{-x} dx =$

$$\begin{bmatrix} u = x^2 & \longrightarrow & du = 2x\,dx \\ v = -e^{-x} & \longrightarrow & dv = e^{-x} dx \end{bmatrix}$$

$$= -x^2 e^{-x} - \underbrace{\int -e^{-x}\, 2x\, dx}_{2\int x e^{-x} dx \,=\, 2[-e^{-x}(x+1)]} =$$

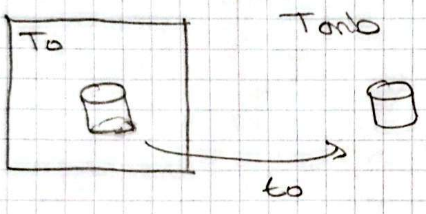
$$= -e^{-x}[x^2 + 2x + 2]$$

Ordinary differential equations (EDOS)

It is an equation that involve functions and its derivatives. they are used to solve dinamic problems

Exeple 1: Termalization

we extract a material out of a fridge with initial temperature T_0 and we expose it to an embironment at T_{amb}. we want to study the T evolution



$$\frac{dT}{dt} = k\,(T - T_{amb})$$

to solve we use the metode of separation of variables: we separate completely the variable T of the variable t

Temperature time

$$\int_{T_0}^{T_f} \frac{dT}{T - T_{amb}} = \int_{t_0}^{t_f} k\,dt$$

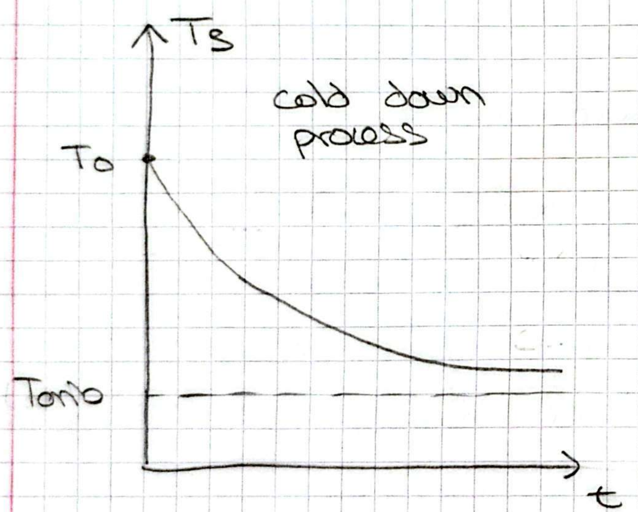
$$\Big[\ln(T - T_{amb})\Big]_{T_0}^{T_f} = \Big[k t\Big]_{t_0}^{t_f}$$

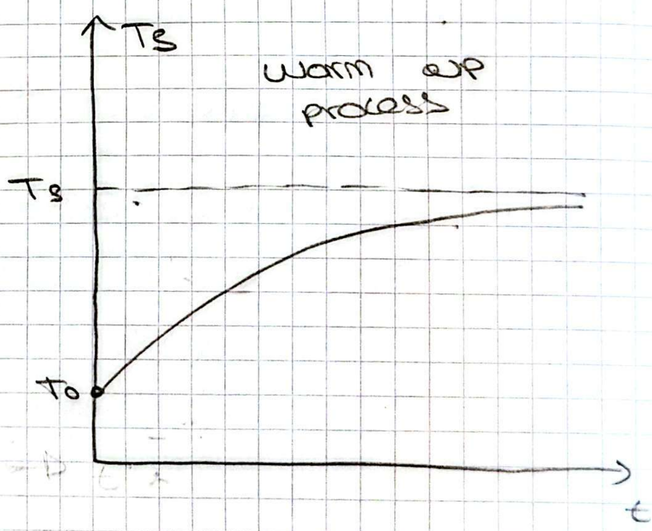
$$\ln \frac{T_f - T_{amb}}{T_0 - T_{amb}} = k\,(t_f - t_0)$$

$$\frac{T_f - T_{amb}}{T_0 - T_{amb}} = e^{k(t_f - t_0)}$$

The temperature evolutions has the usual shape :

$$T_s = (T_0 - T_{amb})\, e^{-\kappa(t_s - t_0)} + T_{amb}$$

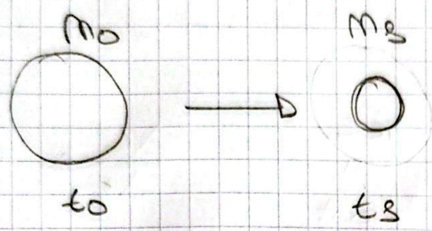




Example 2 : Desintegration

This is used to perform the C-14 test.

A material has an initial amount of instable isotops and it desintegrate with time. We want to know how old is the material by checking the amount of remaining instable material



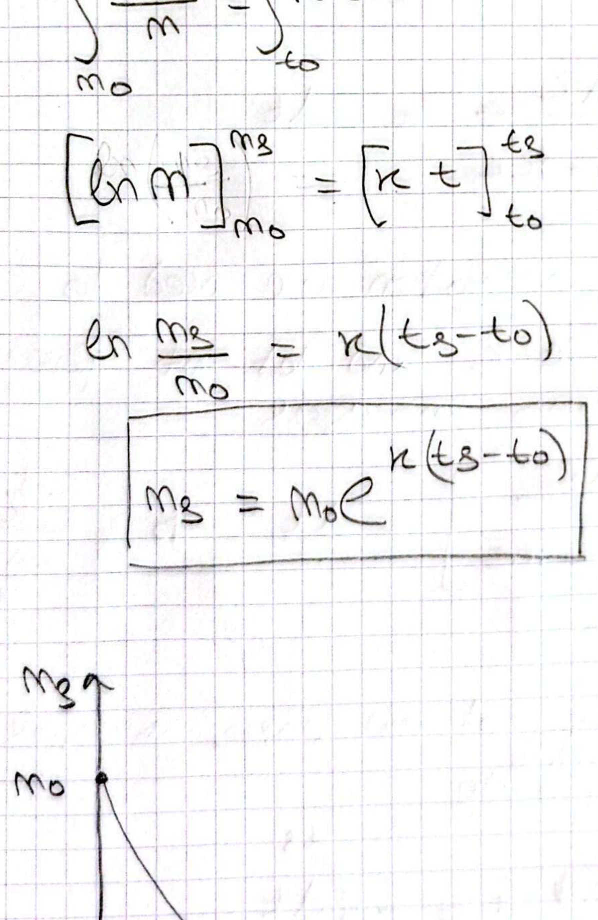
$$\frac{dm}{dt} = k\,m$$

$$\int_{m_0}^{m_s} \frac{dm}{m} = \int_{t_0}^{t_s} k\,dt$$

$$\left[\ln m\right]_{m_0}^{m_s} = \left[k\,t\right]_{t_0}^{t_s}$$

$$\ln \frac{m_s}{m_0} = k(t_s - t_0)$$

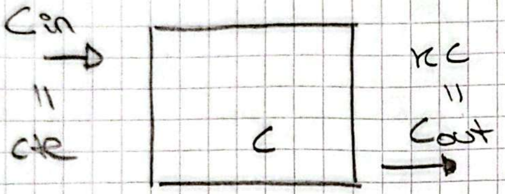
$$\boxed{m_s = m_0 e^{k(t_s - t_0)}}$$



the amount instable material decrease exponentially in time

Exemple 3 : valance :

we introduce and extract sensitives to a body. The rate of extraction is proportional to the internal material. we want to know the amount of internal material as a function of time



$$\frac{dC}{dt} = C_{in} - C_{out}$$

$$\frac{dC}{dt} = A - kC$$

$$\frac{dC}{A - kC} = dt$$

before integration we need to remove the k from the C and let it positive to get an easy integral

$$\frac{-1/k}{-1/k}\left[\frac{dC}{A - kC}\right] = dt \rightarrow \frac{-\frac{1}{k}\,dC}{-\frac{A}{k} + C} = dt$$

$$= 1$$

now we got an easy integral and we can procede

$$\int_{C_0}^{C_f} \frac{dC}{C - \frac{A}{k}} = -k\int_{t_0}^{t_f} dt$$

$$\left[\ln\left(C - \frac{A}{k}\right)\right]_{C_0}^{C_f} = -k\left[t\right]_{t_0}^{t_f}$$

$$\ln\left(\frac{C_S - \frac{A}{k}}{C_0 - \frac{A}{k}}\right) = -k\,(t_S - t_0)$$

$$\frac{C_S - \frac{A}{k}}{C_0 - \frac{A}{k}} = e^{-k(t_S - t_0)}$$

$$C_S = \left(C_0 - \frac{A}{k}\right) e^{-k(t_S - t_0)} + \frac{A}{k}$$